AN INTERPLAY BETWEEN FILTERS AND BOOLEAN TOPOLOGICAL GROUPS

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In studying extremally disconnected topological groups, especially countable ones, there naturally arise various kinds of filters and ultrafilters. Thus, to construct the first consistent example of a nondiscrete extremally disconnected group, Sirota has essentially defined selective ultrafilters on ω and proved their existence under CH.

On the one hand, whenever a countable extremally disconnected group contains a nonclosed discrete set (which amounts to a disjoint family of open sets converging to a point), the very definition of extremal disconnectedness gives rise to a convergent ultrafilter on this set, which is subject to certain constraints imposed by the group structure. In particular, Zelenyuk proved that any such ultrafilter must be what he called "partially selective" (and can be mapped to a P-point ultrafilter). On the other hand, Reznichenko and I have recently proved that any countable topological group in which the identity element has nonrapid filter of neighborhoods contains a discrete set with precisely one limit point. In the case of Boolean groups (which is the only interesting case in the context of extremally disconnected groups, because any extremally disconnected group contains an open Boolean subgroup), the nonexistence of rapid (ultra)filters entails the existence of two disjoint discrete sets for each of which zero is the only limit point; it follows that countable nondiscrete extremally disconnected groups cannot exist in ZFC. However, it is still unclear whether the existence of countable extremally disconnected groups with nonrapid filter of zero neighborhoods is consistent with ZFC.

Naturally, the existence of nonclosed discrete sets satisfying additional algebraic conditions must have even greater consequences. The simplest of such conditions is independence. Let us say that a set X in a Boolean group G is k-independent if $x_1 + x_2 + \cdots + x_k \neq 0$ for any different $x_i \in X$ and that X is independent if it is k-independent for any k, i.e., linearly independent in G considered as a vector space over $\mathbb{Z}_2$. A basis of a Boolean group is any maximal independent set. Any countable Boolean topological group has a closed discrete basis. Of course, it may also have many nondiscrete bases. However, in countable extremally disconnected Boolean groups, closed bases cannot have more than one limit point, and in such groups with nonrapid filter of neighborhoods, all bases are closed and discrete. To be more precise, 4-independent sets in such a group cannot have more than one limit point; all sets which are simultaneously 4- and 6-independent are closed and discrete; and 3-independent sets cannot accumulate to zero.

The last condition turns out to be very interesting in its own right in both countable and uncountable cases. It turns out to be closely related to the so-called 3-arrow ultrafilters. An ultrafilter $\mathcal{U}$ on a set X is said to be κ-arrow if, given any 2-coloring $c\colon [X]^2 \to \{0,1\}$, there exists either an $A \in \mathcal{U}$ such that $c([A]^2) = \{0\}$ or an $F \in [X]^\kappa$ such that $c([F]^2) = \{1\}$. Arrow (and Ramsey) ultrafilters have a natural description in terms of free Boolean topological groups on almost discrete sets. In particular, an ultrafilter is 3-arrow if and only if, in the free Boolean topological group on this ultrafilter, none of the 3-independent sets of elements of length two accumulates to zero. These descriptions are interesting if only because they relate arrow and Ramsey ultrafilters to large sets in groups.

Various notions of large sets in groups naturally arise in dynamics and combinatorial number theory and have numerous applications far beyond these particular fields. Thus, in studying extremally disconnected groups, Reznichenko and I came across a new type of large sets, which we called vast sets. A 3-vast set in a Boolean group is nothing but the complement of a 3-independent set. Thus, 3-arrow ultrafilters are characterized in terms of 3-vast sets, and κ-arrow ultrafilters for other κ are characterized in terms of similar large sets. This makes it possible to distinguish between different, both new and old, types of large sets, which have not yet been distinguished from each other, by using known properties of arrow ultrafilters and free group topology.

Curiously, the technique developed in studying extremally disconnected groups helps (me, at least) to better understand Ramsey and selective ultrafilters. These terms are used interchangeably, but the definitions are different (moreover, there are at least six classical equivalent definitions of selective ultrafilters), and each of them can formally be applied to filters. Clearly, any Ramsey filter is an ultrafilter, but some of the other definitions can be adapted to filters so as to produce classes of filters not necessarily being ultrafilters (such as +-selective filters), while some others cannot. For example, results on the extremal disconnectedness of free Boolean topological groups readily imply that if $\mathcal{F}$ is a filter on ω such that any sequence of elements of $\mathcal{F}$ has a diagonal intersection in $\mathcal{F}$, then $\mathcal{F}$ is a (Ramsey) ultrafilter.

Finally, there still remains the main problem on the existence in ZFC of a nondiscrete extremally disconnected group. Such a group must be uncountable, and proving its existence or nonexistence should most likely involve ultrafilters on uncountable sets. Thus, there arises the need for uncountable generalizations of classical types of filters (first of all, rapid filters) appropriate for the purpose. I have a few ideas and many questions on this topic, which I would like to discuss.

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